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THE SIMPLE ARGUMENT FOR SUBCLASSICAL LOGIC

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Abstract

This paper presents a simple but, by my lights, effective argument for a subclassical account of logic—an account according to which logical consequence is (properly) weaker than the standard, so-called classical account. Alas, the vast bulk of the paper is setup. Because of the many conflicting uses of 'logic' the paper begins, following a disclaimer on logic and inference, by fixing the sense of 'logic' in question, and then proceeds to rehearse both the target subclassical account of logic and its well-known relative (viz., classical logic). With background in place the simple argument—which takes up less than five pages—is advanced. My hope is that the minimal presentation will help to get 'the simple argument' plainly on the table, and that subsequent debate can move us forward.

1. Disclaimer: Logic and Inference

The topic of this *Philosophical Issues* volume is 'philosophy of logic and inference' (where, I presume, the topic is to be parsed *philosophy of: logic and inference*). My philosophy of the relation between logic (qua logical consequence) and inference (qua acceptance-rejection behavior or 'change in view') is that the relation is fairly weak (Beall 2015). Notwithstanding a few differences in detail, I stand largely with the simple picture advanced by Gilbert Harman (1986) long ago: namely, that logical consequence is an *entailment* relation (and monotonic, among other things) while inference is not an entailment relation (or even monotonic); the former works with sentences/propositions while the latter works with mental activities (e.g., rejection, acceptance, etc.). Yes, there is a link that rationality imposes, something to the effect that logical validity constrains what we ought rationally accept and reject—constrains, to some degree, the rationally 'appropriate' change-in-view behavior in which we engage (Beall 2015). But the constraints are very weak, and the link between a logical entailment and *what you ought accept/reject* or *how you ought change your view* is likely to be thin. That logic provides little support for how we ought to 'rebuild our raft while at sea' is epistemologically frustrating; but that's how things are, at least from my perspective. In ways that become clearer in the discussion below: what logic (qua logical consequence) does is both firmly mark the remotest boundaries of theoretical possibilities and also furnish the weakest skeletal structure of our true (closed) theories. And that's pretty much it.

There are many important questions about the link between logic and inference, and how to correctly formulate the link. I have little to offer on those questions here.[1] What I do offer here is indirectly related to the issue of 'linking' logic and inference. In particular, I aim to present, in a full-enough but concise way, a simple argument for a particular subclassical account of logical consequence. This is at least indirectly related to the link between logic and inference: at the very least, the sheer weakness of logic makes it plain that much of our rational inferential behavior is in many (most?) cases phenomenon-/theory-specific, unlike logic, which is universal.[2]

Disclaimer done. What follows is (alas, lengthy) setup for the target simple argument—which appears in §4—for a subclassical, strictly weaker-than-classical account of logic.

2. The Role of Logical Consequence: Closure

There are many formal entailment relations over any given natural language, and equally so over any fragment of natural language serving as the language of a particular theory, where a formal entailment relation is an absence-of-counterexample relation whose valid 'forms' are defined via some given stock of expressions (in the usual way).

Not only are there many formal entailment relations over any given language (of any interest); there are also many roles that any given entailment relation might play, but some such roles are realized by exactly one such relation. *Logical consequence*, on the picture discussed in this paper, is like that: it's the formal entailment relation that (uniquely) plays a particular role. What role? The answer is a very traditional and very familiar one. Logical consequence is the formal entailment relation that plays the role of *universal closure relation*—or *universal basement-level closure relation*—involved in all of our true theories (in particular, in the closure relations of our true theories).[3] This account of logical consequence (qua universal consequence/closure) is best understood via a common picture of theories and the two-fold task of theorists.

2.1. Theories and the twofold task of theorists

The concern throughout is with truth-seeking disciplines, and in particular the goal of true theories. Here, once she has identified her target phenomenon (about which she aims to give the true and as-complete-as-possible theory), the task of the theorist is twofold:

- gather the truths about the target phenomenon
- construct the right closure relation to 'complete' the true theory—to give as full or complete a true theory as the phenomenon allows.

This twofold task is as basic as it is familiar—and, of course, in no way novel. This is just what we do as truth-seeking theorists, whether we are in mathematics (even if we don't quite know what 'makes true' the theories), physics, theology, biology, philosophy, and more—for every phenomenon that contributes to the overall makeup of reality. Once our target phenomenon has been identified (enough to get on with business, so to speak) we then search and gather whatever truths we can; and after that we aim to give the right closure relation for the theory, all with the aim of giving as full/complete an account of the target phenomenon as possible.

On this picture, we think of the theories as pairs $\langle T, \vdash_T \rangle$ where $\vdash_T$ is the closure relation for T, but ultimately, as is common in logical studies, we identify the theory T with the closed theory—the stock of truths closed under $\vdash_T$.

This picture of the theorist's twofold task—and the corresponding picture of a 'closed theory' (closed under the given closure/consequence operator)—is the one in which logic's role is plain.

2.2. Where is logic in this picture?

Logic is the basement-level closure/consequence relation involved in all of our true theories, where our true theories are pictured as pairs (to highlight the closure relation):

$$\langle T_1, \vdash_{T_1} \rangle, \langle T_2, \vdash_{T_2} \rangle, \langle T_3, \vdash_{T_3} \rangle \ldots, \langle T_n, \vdash_{T_n} \rangle$$

Logic shows up in each such theory-specific consequence relation $\vdash_{T_i}$; it is the relation under which all true theories, so understood, are closed; it is the relation on top of which all closure relations for our true theories are built.

2.3. The second task: extra-logical consequence

The theorist, as above, has a twofold task: gather the truths and, second, construct the right closure relation for the theory. But why the second task in building her theory if logical consequence is already there—already 'built' and present, so to speak? The answer, of course, is that in most (all?) cases the full/complete truth of the target phenomenon requires a closure relation that goes well beyond the basement-level consequence relation of logic.

Examples are ubiquitous. Take the true (and complete-as-possible) theory of knowledge. Here, one's target phenomenon (viz., knowledge and/or what is known) demands a theory whose stock of special non-logical vocabulary involves (let me say for simplicity if not entirely for accuracy) a knowledge operator $\mathbb{K}$ (viz., in English, 'it is known that...'). Now, logic demands of knowledge claims (for present purposes, claims of the form $\mathbb{K}A$) what it demands of all claims in the language (or in any language): namely, that they interact with the logical vocabulary just so (where the 'just so' is spelled out by what follows logically from what, etc.). In particular, where p is logically atomic (i.e., has no logical vocabulary), since logic does not discriminate between $\mathbb{K}p$ and p, logic itself counts as invalid the form $\mathbb{K}p \therefore p$. (And it should. There are staggeringly many counterexamples recognized by logic.) But our true theory of knowledge and/or what is known requires the entailment from $\mathbb{K}p$ to p.[4] And this is what the theorist's second task involves: she needs *not* to build the logical part of the theory's consequence/closure relation; that part is indeed already built and present (so to speak). What the theorist's second task involves is the construction of the right extra-logical consequence/closure relation that serves to complete (as far as possible) the true theory.

On the going (and I hope very familiar) picture, logic is in all of our true theories; it's part of the theory's closure relation—and almost always a proper part. The theorist's second of two tasks begins on the foundation of logical consequence, and builds a theory-specific closure relation on top of that foundation.

2.4. What is logical vocabulary in this picture?

Logical vocabulary, in the target picture, is the vocabulary that figures in all of our true—and complete-as-possible—theories.

Defining 'logicality' is notoriously difficult, and I have little (if anything) new to add. Debates about what counts as logical vocabulary are often seen—perhaps with a background polaroid of Carnap saying 'in logic there are no morals'—as unfruitful (to put a gentle spin on the matter). I agree with this sentiment in many ways. On the other hand, debates between those who say that logic is (sub-) classical are not pointless and are not 'merely terminological'. There's something important being debated. On the current picture, what's being debated is the relation that plays the basement-level closure relation in our theories. And if that's right, then there's some stock of vocabulary that figures in the given basement-level relation (in particular, in the relation's valid 'forms'). While I have no knockdown argument on the matter, it strikes me that the the stock of vocabulary common to all true (and complete-as-possible) theories is what tradition has often put forward: namely, the standard stock of first-order vocabulary (shy of identity predicates and function signs, both of which have often been seen to carry a bit too much 'substance' to be 'properly logical'). And that's the vocabulary that, at least for present purposes, I take to be logical—not because I have a useful independent criterion (e.g., some twist on invariance or etc.) but rather because the role of logic qua universal closure relation naturally suggests as much.[5]

2.5. Trinity of traditional features

One thing that jumps out of this picture of logic—and perhaps what explains its familiarity—is that it immediately reflects the trinity of traditional features associated with logical consequence.

2.6. Logic is universal

That logical consequence is *universal* is a familiar and traditional claim. In what way is logic universal on the going picture? Straightforwardly and obviously: it matters not at all to which (true and complete-as-possible) theory T_i we turn in the long parade of such true theories; logic is involved in all of them. No matter where in reality our true theories are directed, logic is involved.

2.7. Logic is topic-neutral

Another core feature of logic is *topic-neutrality*. Similar to universality, the topic-neutrality of logical consequence on the going picture is plain: it matters not one bit whether the topic of your true (and complete-as-possible) theory is tractors, snakes, triune gods, or the halting problem; logic applies equally to all topics—full stop. Whatever is logically valid for topic X is logically valid for every topic Y; and whatever is logically invalid for topic X is logically invalid for every topic Y. A better slogan than Carnap's is that *in logic there is no discrimination* of topics: whatever is logically invalid in one theory is logically invalid everywhere.

2.8. Logic is intransgressible

Finally, filling out the trinity of traditional features, logic has long been said to be *intransgressible* (though perhaps that word hasn't been around as long as equivalent formulations of the slogan). In a slogan: you can't transgress logic. How so? The picture at hand makes it plain: if you're advancing a true (and complete-as-possible) theory then your theory 'obeys logic'. After all, logic is involved at the bottom level of every such true theory.

2.9. Logic, in summary

Summary: think of logic as the relation (of logical consequence) that plays the traditional universal-closure role. When philosophers debate whether logic is (sub-) classical they are—or, at least in my view, perhaps should be—debating (only) which consequence relation plays the universal closure role.

3. FDE and the Standard Account

Think of logic as entailment—absence of counterexample—in virtue of 'logical form' (where logical form is specified per usual via logical vocabulary). Then logic recognizes a space of possibilities, each of which serves as a potential counterexample to any 'argument' (think set-sentence pairs). The difference between the standard (so-called classical) account and subclassical accounts is that the former recognizes a smaller space of possibilities—potential counterexamples—than the latter. On the subclassical picture, logic rules fewer possibilities out than on the standard picture.

3.1. The FDE account of consequence

Turn to the FDE (for 'first-degree entailment') account first.[6]

3.1.1. Syntax

The syntax is the usual stock of logical vocabulary, namely, the first-order vocabulary without any logical predicates (hence, no identity) and without any function signs:

- Boolean quartet:[7]
  - unary: truth operator (†), generating *nullations*
  - unary: falsity operator (¬), generating *negations*

  - binary: conjunction ($\wedge$) generating *conjunctions*
  - binary: disjunction ($\vee$) generating *disjunctions*

- Quantifiers:

  - universal: ($\forall$) generating *universals*
  - existential: ($\exists$) generating *existentials*

- Variables: 'object variables' (x, y, z with (or -out) subscripts $i \in \mathbb{N}$)
- A-logical vocabulary: parentheses (punctuation)

This is the full stock of logical (cum a-logical) vocabulary, as it shall be understood here.

The definitions of *wff* and *sentence* are per the usual recursive account.

3.1.2. Semantics

Logical consequence is an absence-of-counterexample (set-sentence) relation, a formal entailment relation where the 'forms' are given by the logical vocabulary.[8] The counterexamples are possibilities that logic recognizes; and these possibilities are modeled by certain kinds of 'points' or 'models' (which, in turn, are generally set-theoretic constructions).

The recipe for constructing models is standard: start with 'interpretations' (which, together with variable assignments, can serve as 'points' for evaluation of sentences); pare down the space of points to just the ones that respect the truth/falsity conditions (including the space of semantic values) for logical vocabulary; and then let the resulting space represent logic's space of (the models of the) possibilities—the would-be counterexamples—recognized by logic.

Interpretations

Interpretations are pairs $I = \langle D, d \rangle$ where $D \neq \emptyset$ and d is a denotation function that assigns each constant c in the language (of a given theory) an element $d(c)$ of D, and d also assigns each n-ary predicate P a pair $\langle P^+, P^- \rangle$, where each of P^+ and P^- are subsets of D^n. (Intuitively, P^+ is the *extension* of P, that is, the set of all objects in D of which P is true according to the given interpretation; and P^- is the *antiextension*, that is, the set of all objects in D of which P is false according to the given interpretation.)

As usual, given a *variable assignment* v, standardly understood, there's a (total) denotation function δ induced by the interpretation's denotation function d, namely, $\delta(t) = v(t)$ if t is variable, and otherwise $\delta(t) = d(t)$.

Points: these interpretations-cum-variable-assignments serve as 'points' at which sentences (generally, wff) of the language (of a theory) get 'semantic values' or 'values'. What values?

Values

The space of (semantic) values is broader than the Standard pair $\{1, 0\}$ by two. In particular, following Dunn's (1966) approach, we take the space of logic's possibilities (for any sentence) to be modeled by the powerset $\wp(\{1, 0\})$ of $\{1, 0\}$, with each element taking a suggestive name:

$$\mathsf{t} = \{1\}$$

$$\mathsf{f} = \{0\}$$

$$\mathsf{b} = \{1, 0\}$$

$$\mathsf{n} = \emptyset$$

One can think of t and f as modeling the standard idea of a sentence being (just) true and (just) false, respectively, while the additional values (viz., b and n) represent broader possibilities of a sentence being glutty (i.e., a glut of truth and falsity) and a sentence being gappy (i.e., a gap of truth of falsity). This way of thinking of the values is reflected in standard truth/falsity conditions.

Truth and falsity conditions

Our 'points', as above, are interpretations together with variable assignments. These serve as points at which sentences in the language (of a theory) may be evaluated. But not all such points—because not all such interpretations—are relevant to logical consequence; the truth/falsity conditions for logical vocabulary serve to fix the target points that logic recognizes as 'possibilities'—as 'candidate counterexamples' for logical-consequence claims.

Towards precision let our *points* officially be pairs $\langle I, \nu \rangle$ of interpretation functions and variable assignments. As usual, we want to talk about the *semantic value* or *semantic status* of a wff (of the given language) at points. Towards this end we let $|A|_{\langle I, \nu \rangle}$ be the value of wff A at point $\langle I, \nu \rangle$, where the range of this semantic-value function is our set of values $\{\mathsf{t}, \mathsf{f}, \mathsf{b}, \mathsf{n}\}$.

The *truth/falsity conditions for the logical vocabulary* are exactly the standard truth/falsity conditions, where, following Belnap's (1977) terminology, '$1 \in |A|_{\langle I,\nu\rangle}$' and '$0 \in |A|_{\langle I,\nu\rangle}$' may be read as that A is at least true (respectively, at least false) at the given point.[9]

- Atomics: let P be an n-ary predicate and $t_1, \ldots, t_n$ be n terms. Then
  - $1 \in |Pt_1, \ldots, t_n|_{\langle I,\nu\rangle}$ iff $\langle \delta(t_1), \ldots, \delta(t_n)\rangle \in P^+$.
  - $0 \in |Pt_1, \ldots, t_n|_{\langle I,\nu\rangle}$ iff $\langle \delta(t_1), \ldots, \delta(t_n)\rangle \in P^-$.
- Unary boolean: let A be any wff.
  - Truth: $1 \in |\dagger A|_{\langle I,\nu\rangle}$ iff $1 \in |A|_{\langle I,\nu\rangle}$.
  - Truth: $0 \in |\dagger A|_{\langle I,\nu\rangle}$ iff $0 \in |A|_{\langle I,\nu\rangle}$.
  - Falsity: $1 \in |\neg A|_{\langle I,\nu\rangle}$ iff $0 \in |A|_{\langle I,\nu\rangle}$.
  - Falsity: $0 \in |\neg A|_{\langle I,\nu\rangle}$ iff $1 \in |A|_{\langle I,\nu\rangle}$.
- Binary boolean: let A and B be any wff.
  - Conjunction: $1 \in |A \wedge B|_{\langle I,\nu\rangle}$ iff $1 \in |A|_{\langle I,\nu\rangle}$ and $1 \in |B|_{\langle I,\nu\rangle}$.
  - Conjunction: $0 \in |A \wedge B|_{\langle I,\nu\rangle}$ iff $0 \in |A|_{\langle I,\nu\rangle}$ or $0 \in |B|_{\langle I,\nu\rangle}$.
  - Disjunction: $1 \in |A \vee B|_{\langle I,\nu\rangle}$ iff $1 \in |A|_{\langle I,\nu\rangle}$ or $1 \in |B|_{\langle I,\nu\rangle}$.
  - Disjunction: $0 \in |A \vee B|_{\langle I,\nu\rangle}$ iff $0 \in |A|_{\langle I,\nu\rangle}$ and $0 \in |B|_{\langle I,\nu\rangle}$.
- Quantifiers: let A be any wff, and u any variable.[10]
  - Univ: $1 \in |\forall u A|_{\langle I,\nu\rangle}$ iff $1 \in |A|_{\langle I,\nu^{[u]}\rangle}$ for each u-variant $\nu^{[u]}$ of ν.
  - Univ: $0 \in |\forall u A|_{\langle I,\nu\rangle}$ iff $0 \in |A|_{\langle I,\nu^{[u]}\rangle}$ for some u-variant $\nu^{[u]}$ of ν.
  - Exist: $1 \in |\exists u A|_{\langle I,\nu\rangle}$ iff $1 \in |A|_{\langle I,\nu^{[u]}\rangle}$ for some u-variant $\nu^{[u]}$ of ν.
  - Exist: $0 \in |\exists u A|_{\langle I,\nu\rangle}$ iff $0 \in |A|_{\langle I,\nu^{[u]}\rangle}$ for each u-variant $\nu^{[u]}$ of ν.

Note that these are exactly the truth/falsity conditions involved in the standard (classical) account of consequence. The salient difference is that in the standard account the falsity clauses are often omitted; they're omitted not because they're rejected by classical-logic theorists; they're omitted because, in the narrow constraints imposed by the standard account of consequence, the falsity clauses turn out to be redundant. Still, the foregoing truth/falsity conditions remain—notwithstanding said omission issues—exactly what is involved in the standard account.

Parenthetical remark. The FDE semantics are very simple and natural. As Anderson and Belnap (1975) and Anderson, Belnap, and Dunn (1992) elaborated, if one prefers a sort of simple 'algebraic' approach, one can give the same semantics reflected in the foregoing truth/falsity conditions by instead thinking of the four values as ordered by the following diamond ('lattice'):

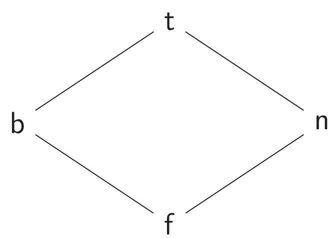

Figure 1. 'Four corners of truth'

Logic's truth (nullation) operator is fixed at all values/nodes; the falsity (negation) operator toggles t and f and is fixed at each of b and n; and the conjunction and disjunction operators are, as usual, greatest lower bound (infimum) and least upper bound (supremum), respectively. In turn, variable assignments to the side, one can think of the quantifiers as simply mimicking or mirroring the conjunction (universal) and disjunction (existential) operators. While the quantifiers play a key role in the target FDE account of logic as universal closure (particularly with respect to 'shrieking' and 'shrugging', presented below), any readers new to the subclassical account might find it easier to simply focus on the natural and simple propositional case in this lattice-centered picture. *End remark.*

3.1.3. Models and consequence

Logical consequence is an absence-of-counterexample relation. The counterexamples are modeled by models.

Models

Models in FDE are simply points (per §3.1.2 'points') that obey the truth/falsity conditions (per above). In other words, models are simply interpretations in which all of the logical vocabulary behave per the standard truth/falsity conditions.

Logical consequence

Logical consequence is a set-sentence relation, where the set contains (only) sentences.[11]

Definition 1 (Counterexample). *An FDE model m is a* counterexample *to $X \therefore A$ (or, simply, $\langle X, A \rangle$) iff everything in X is at least true-in-m but A fails to be even at least true-in-m, that is:*

- *everything in X contains 1 (i.e., is either* t *or* b*)*
- *A does not contain 1 (i.e., is either* f *or* n*).*

Definition 2 (Consequence). *X logically entails A (or A is a logical consequence of X) iff there's no counterexample to $X \therefore A$.*

Definition 3 (Notation: Logic). *$X \vdash A$ (with no subscript on the turnstile) iff X logically entails A.*

Comparison of the just-given FDE account of logical consequence with the Standard (classical) account is given in §3.3 following a rehearsal of said standard account.

3.2. The Standard/Classical account of consequence

The standard account of logical consequence (qua universal closure) is very, very similar to the FDE account; the difference is that the former narrowly constrains the space of possibilities recognized by logic. The standard account runs as follows.

3.2.1. Syntax

This is the same as in FDE.

3.2.2. Semantics

There are two natural and equivalent ways to present the semantics vis-a-vis FDE. I follow the standard route (but flag the other route in passing).

Interpretations (and points)

These are (both) the same as in FDE except for a constraint on the denotations of predicates. In particular, interpretations are pairs $I = \langle D, d \rangle$ where $D \neq \emptyset$ and d is a denotation function that, as in FDE, assigns each constant c in the language (of a given theory) an element $d(c)$ of D, and d also assigns each n-ary predicate P a pair $\langle P^+, P^- \rangle$, where each of P^+ and P^- are subsets of D^n *with the following constraints*:

- Predicate Exhaustion: $P^+ \cup P^- = D^n$.
- Predicate Exclusion: $P^+ \cap P^- = \emptyset$.

While FDE allows for interpretations in which either Predicate Exhaustion or Predicate Exclusion (or both) fail (while also allowing for ones that completely obey such constraints), the Standard (classical) account rules them out, and hence rules out (for example) an object's being in both the extension and antiextension of a predicate, and also rules out an object's being in neither the extension nor antiextension of a predicate.

Accordingly, the *points* (of evaluation) recognized by the standard account of logical consequence (viz., classical) are interpretation-cum-variable-assignment pairs where the two predicate constraints are in force.

Values

The space of values is strictly smaller than the space $\wp(\{1, 0\})$ recognized by FDE; the space is just $\{\mathsf{t}, \mathsf{f}\}$, where $\mathsf{t} = \{1\}$ and $\mathsf{f} = \{0\}$.[12]

Truth and falsity conditions

These are exactly the same as in FDE.[13]

3.2.3. Models and consequence

(Classical/Standard) Models

The *classical models* are just the FDE models that respect the strictly smaller space of values—or, equivalently, respect the predicate-exhaustion and predicate-exclusion constraints.

Logical consequence

Consequence, just as in FDE, is absence of counterexample, where the relation is a set-sentence relation.

Definition 4 (Classical counterexample). *A model m is a* classical counterexample *to $X \therefore A$ (or, simply, $\langle X, A\rangle$) iff everything in X is (at least) true-in-m but A fails to be (even at least) true-in-m, that is:*

- *everything in X contains exactly 1 (and, hence, given the definition of* classical models*, is* t*)*
- *A does not contain 1 but contains something (hence, given the definition of* classical models*, is* f*).*

Definition 5 (Classical consequence). *X logically entails A (or A is a logical consequence of X) iff there's no classical counterexample to $X \therefore A$.*

Comment on notation: because this paper argues that the FDE account of logic is the right account (at least vis-a-vis the Standard account) the notation '$X \vdash_c A$' will be used for classical consequence where (if anywhere) context fails to make the matter plain.

3.3. Some salient similarities and differences

Here are some (not independent) salient similarities and differences in the two accounts of logic qua universal closure.

3.3.1. Logical truths

Which sentences, if any, are logically true (i.e., at least true-in-m for every possibility or model m recognized by the given relation of logical consequence)? Equivalently: which sentences, if any, are logical consequences of the empty set—and, hence, in each and every (closed) true theory?

FDE: none

According to the FDE account of logic there are no logical truths. Witness: there's an FDE model according to which every sentence is gappy. (First step: let every predicate P in the language get the denotation $d(P) = \langle \emptyset, \emptyset \rangle$, that is, the 'anti-trivial' or 'completely gappy' interpretation. Induction will tell that every sentence of the given interpretation is similarly gappy.)

Standard: usual

According to the Standard account (viz., classical logic) there are logical truths, namely, the usual ones canvassed in standard introductions to (classical) logic.

Comment on the logical-truths front

FDE scores high in this regard, inasmuch as logic (qua topic-neutral universal closure relation on all true theories) leaves it to *theories* to say what is true; logic on its own contributes no truths whatsoever to true theories.

3.3.2. Deduction theorem for logic's conditional

Logic's vocabulary is sparse, perhaps strikingly so with respect to conditionals. The material conditional, defined (as usual) via logic's disjunction and falsity connectives, viz.,

$$A \supset B \ := \ \neg A \vee B$$

is logic's principal conditional—that is, the principal conditional expressed in logical vocabulary. The question at hand is whether this conditional exhibits so-called deduction-theorem behavior with respect to logic (i.e., with respect to logical consequence). The question is twofold, each fold concerning one direction of the usual deduction-theorem pattern, where $\vdash$ is logical consequence:

- If $\{A_1, \ldots, A_n\} \vdash B$ then $\vdash A_1 \wedge A_2 \wedge \ldots \wedge A_n \supset B$.
- If $\vdash A_1 \wedge A_2 \wedge \ldots \wedge A_n \supset B$ then $\{A_1, \ldots, A_n\} \vdash B$.

FDE: no-'yes'

On the FDE account of logic, the dt2 direction holds (only) vacuously (e.g., see §3.3.1).[14] On the other hand, the FDE account refutes the given deduction-theorem pattern along the dt1 direction. For example, note that $A \vdash A$ but, since there is a model in which $A \supset A$ is gappy (i.e., has value n in the model), $\nvdash A \supset A$. (There are many other examples.)

Standard: yes-yes

The standard account has it that logic's conditional satisfies both directions of the given deduction-theorem behavior.

Comment on the deduction-theorem front

One might think that deduction-theorem behavior is important in many of our true theories—even for logic's material conditional. (I think as much.) But whether logical consequence alone should satisfy deduction-theorem behavior for logic's vocabulary is far less plausible. On the Standard account the deduction theorem holds and thereby infuses the logical truth of various material-conditional claims with a wisp of truth about logical consequence itself. On that picture, one need merely look at the intersection of all true theories to read off logical consequence from the material conditionals in said intersection. But, then, on that picture, there's a sense in which logical consequence talks about itself in every true theory—maybe not explicitly (since one needs to see all other true theories), but loudly nonetheless.

A better picture is given by the FDE account. On that picture logic says nothing at all—even when, per the dt1 direction, according to it (viz., logic) some claim follows from some claim(s). On the FDE picture logic, qua universal closure relation, does not on its own—in its own vocabulary—give even a wisp of the truth about itself in any way; a *theory* of logic will do that, where one might (or might not) expect some deduction-theorem behavior arising from theory-specific constraints or vocabulary. For example, just as in the true theory of arithmetic (say, classically closed Peano Arithmetic), where the theory's closure relation involves some deduction-theorem-like behavior with logic's conditional (viz., the material conditional defined via logic's falsity and disjunction connectives), so too we might have that in the true theory of logic—i.e., of logical consequence (viz., FDE)—we may have a conditional in the language of the theory which reflects some deduction-theorem-like behavior. But logic itself—as a relation (viz., universal consequence)—directly says (implies, delivers, etc.) no truths whatsoever about logical consequence; that's for a theory of logic.

3.3.3. Truth/Falsity connectives

An important question for any account of logic (qua universal closure) concerns what, if any, constraints the account takes logic to impose on logic's truth (nullation) and falsity (negation) connectives. FDE and the standard account give very different answers.

FDE: no constraints

Without loss of generality, focus only on atomic sentences p. FDE imposes no constraints on $\dagger p$ and $\neg p$.

Standard: 'classical' constraints

Again, focusing just on atomic sentences p, the standard (classical) account imposes *both* 'exhaustion' and 'exclusion', where these jointly make up the *classical constraint* (on $\dagger$ and $\neg$):

- Exhaustion: exactly one of $\dagger p$ and $\neg p$ is (at least) true in every possibility (potential counterexample) recognized by logic.
- Exclusion: not both of $\dagger p$ and $\neg p$ are (at least) true in any possibility (potential counterexample) recognized by logic.

Given the (given) compositional nature of the (semantics of the) logical vocabulary, the classical constraint immediately generalizes to all sentences—not just atomics. (In fact, the classical constraint is reflected in the predicate-level constraints on classical interpretations per §3.2.2.)

Comment on truth/falsity constraints

FDE scores high on its neutrality with respect to the classical constraint (exhaustion and exclusion). Logic (qua universal closure on true theories) is topic-neutral, leaving the theorist to figure out whether the phenomenon in question (at the center of the theory) is such that some predicate (in the language of the theory) is true of it, false of it, neither true nor false of it, or both true and false of it. That many true theories wind up ruling out the latter two options is not a strike against the topic-neutrality of logic's recognizing such options. Ruling such options out of logic's space of options requires an extra (and arguably ad hoc) imposition of exclusion and/or exhaustion.[15] (I briefly return to the ad-hoc-appearance consideration in §4.3.)

3.3.4. Salient validities: De Morgan core

Both FDE and the Standard account agree on the centrality of familiar De Morgan behavior such as

$$\neg(A \vee B) \dashv\vdash \neg A \wedge \neg B$$

and so on.[16] But there is a difference between the two accounts: FDE says that *De Morgan behavior is all that there is* to logic's demands; the Standard account claims—as discussed above—that there is much else that logic demands (including, e.g., truths of various sorts, etc.).

Comment on De Morgan core

Both the FDE and Standard accounts score well here, with De Morgan behavior traditionally seen as one of the core traits of logic qua universal closure. The question of whether the Standard account does better than FDE by claiming that logic makes more demands than just De Morgan patterns is an issue to which 'the simple argument' of this paper is in part directed.

3.3.5. Salient invalidities: usual and beyond

Both FDE and the Standard account take the standard stock of logically invalid forms—that is, the classically invalid forms—to be just that: invalid according to logic. But FDE's list of invalid forms goes beyond that.

FDE: detachment and more

In addition to the invalidity of classically valid forms whose (classical) validity turns only on the existence of logical truths (e.g., $A \vdash B \vee \neg B$ etc.) the FDE account of logic maintains that there are other salient invalidities, notably:

- Detachment: $A, A \supset B \nvdash B$.[17] (Counterexample: let A be a glut and B either just false or a gap.)
- Explosion: $A \wedge \neg A \nvdash B$. (Counterexample: same as above.)
- Dual of Explosion (viz., Excluded Middle):[18] $B \nvdash \neg A \vee A$. (Counterexample: let B be just true or glutty; let A be gappy.)

Since it recognizes no gluts or gaps—precisely the possibilities required for the foregoing counterexamples—the standard account of logic claims that, contra FDE, each of these forms is valid.

Comment on salient invalidities

Many philosophers see the logical *in*validity of any classically valid forms as a devastating bug of the FDE account of logic (qua universal closure relation for all true theories). This view is mistaken. That this is so is at least partly illustrated by the simple argument for FDE over the Standard account—to which we (finally) turn.

4. The Simple Argument for FDE

The main aim of this paper is to present, as concisely as possible, the simple argument for the FDE account over the Standard account. The argument is indeed simple and, by my lights, effective: we lose nothing by accepting the FDE account; and we gain something (valuable) by doing so. The argument, slightly filled out, runs as follows.[19]

4.1. We lose nothing: all true classically closed theories

By accepting the FDE account we lose what would otherwise be *logically backed* entailments; but this needn't be—and in fact isn't—a loss of true classically closed theories. After all, most of our true theories, including all of the classically closed theories that have vocabulary beyond the logical vocabulary (viz., all of them), have closure relations that involve *logically* invalid—or at least not-logically-valid—entailments, and this for the simple reason that logical entailments involve only the tiny stock of logical vocabulary. (Compare §2.3.) So, losing logically backed entailments in our theories is not a loss of true classically closed theories.

What we 'lose', of course, is the mistaken thought that logic precludes a vast portion of the space of otherwise theoretical possibilities at our theory's disposal; those logical possibilities are theoretical possibilities unless we—qua theorists of the target phenomenon—rule them out as theoretical *im*possibilities. This is more work epistemically than if logic were to do the ruling out; but the metaphor of rebuilding our raft while at sea is rightly famous because, at least in theory building (and change-in-view behavior generally), it's apt. Epistemology aside, the work of treating some of logic's space of possibilities as theoretical impossibilities is both common and straightforward; it takes the explicit form of what I call 'shrieking' and 'shrugging' of theories.

4.1.1. Shrieking theories

Definition 6 (shrieked predicate). *To* shriek a predicate *P in the language of theory T one imposes the following condition on T's closure relation $\vdash_T$, where $\perp$ is true in no models of T:*[20]

$$\exists x(Px \wedge \neg Px) \vdash_T \perp$$

Imposing this condition on a theory's closure/consequence relation has the effect of reducing the space of logical possibilities with respect to predicate P to only non-glutty ones, thereby treating the glutty 'options'—recognized by logic—as theoretical *im*possibilities.

Definition 7 (partially shrieked theory). *To* partially shriek a theory *is to shriek some predicates in the language of the theory.*

Definition 8 (shrieked theory). *To* shriek a theory *is to shriek all predicates in the language of the theory.*

Fact 1. *If T is shrieked then material detachment (i.e., detachment using logic's material conditional) is valid according to the theory (i.e., according to the theory's closure/consequence relation).*[21]

4.1.2. Shrugging theories

This is more familiar from so-called paracomplete accounts of logic. In the case of FDE shrugging—the sort of dual of shrieking—takes the form of shrug conditions.

Definition 9 (shrugged predicate). *To* shrug a predicate *P in the language of theory T one imposes the following condition on T's closure relation* $\vdash_T$*, where* $\top$ *is true in all models of T:*[22]

$$\top \vdash_T \forall x(Px \vee \neg Px)$$

Imposing this condition on a theory's closure/consequence relation has the effect of reducing the space of logical possibilities with respect to predicate P to only non-gappy ones, thereby treating the gappy 'options'—recognized by logic—as theoretical *im*possibilities.

Definition 10 (partially shrugged theory). *To* partially shrug a theory *is to shrug some predicates in the language of the theory.*

Definition 11 (shrugged theory). *To* shrug a theory *is to shrug all predicates in the language of the theory.*

Fact 2. *If T is shrugged then material identity (i.e., identity using logic's material conditional) is valid according to the theory (i.e., according to the theory's closure/consequence relation).*[23]

4.2. We gain something: live options for true theories

As above, we don't lose any true theories, including the many true classically closed theories. But what by way of theories do we gain? The answer is clear: we gain the possibility of true glutty theories and true (and prime) gappy theories.[24] These possibilities mightn't be strikingly important in the face of normal phenomena with which natural science or even mathematics deals; however, they are strikingly important for the strange phenomena at the heart of other theories (e.g., paradoxes, weird metaphysical entities, more).

That we gain something (valuable) is not backed just by the observation that theories of abnormal phenomena might benefit from having glutty and gappy options that logic (qua FDE) recognizes. That we gain something valuable is backed by at least two very clear considerations.

4.2.1. Prima facie gluts and gaps

There are some phenomena that wear gluttiness on their face, and some that wear gappiness on their face; they just appear, prima facie, to be glutty or gappy. Obvious witness: liar sentences appear to be strangely twisted phenomena that are 'overdetermined' or otherwise glutty—'I am not true', etc. They simply *look* that way; that's why they pop out as hard, contradictory paradoxes. On the other hand, truth-tellers (or other forms of apparent 'indeterminacy') appear to be 'underdetermined' or otherwise gappy—'I am true'. They simply *look* that way; that's why they pop out as hard but non-contradictory paradoxes.[25]

4.2.2. Current candidates on the table

There are well-known and live-option FDE-based (including extensions of FDE) theories that are—in fact and not just in principle—candidates for the truth (i.e., true theory) of strange phenomena (e.g., liars, truth-tellers, properties, gods, and more) (Beall 2008; Field 2008; Maudlin 2004; Priest 2006). Such theories are not merely drawings on the walls of conceptual space; they are current contenders—genuinely live-option candidates. They can be at best treated as mere doodles in conceptual space by the Standard account of logic, contrary to the truth of their status.

4.2.3. Putting the pieces together

The two considerations in §4.2.1–§4.2.2 alone would be insufficient to motivate the FDE account of logic (qua universal closure) if the account resulted in a loss of any true theory; but—as per §4.1—no loss results. Given the absence of loss, the two given considerations motivate an adoption of the FDE account of logic (qua universal closure)—or at least a subclassical account over the classical account.[26]

4.3. Bonus argument: ad hocery of standard account

That there's not only no loss of true theories but also gain of strong candidates for true theories is strong reason to adopt the FDE account of logic over the standard account. Another reason concerns apparent ad hocery of the standard account.

Consider logic's truth and falsity operators, ignoring the former as usual (because logically redundant). Here are the standard truth-in-x and falsity-in-x conditions for logic's falsity operator (logical negation), where x may be thought of as a possibility recognized by logic, where these in turn get modeled by precise models (usually some set-theoretic construct) or some similar sort of 'point' at which sentences are evaluated.

- $\neg A$ is true-in-x iff A is false-in-x.
- $\neg A$ is false-in-x iff A is true-in-x.

No theorist in the subclassical family or up through the classical camp rejects these truth/falsity conditions. Those in the classical camp, who claim that logic (qua universal closure) is per the standard account, see the falsity clause (i.e., the second condition above) as superfluous; but it's not something that is or otherwise should be rejected.

The pressing question: why think that the falsity clause (above) is superfluous? The answer, of course, is that those in the classical camp impose two constraints on logic's possibilities—on the various points x in the given *true-in-x* and *false-in-x* conditions. In particular, the classical camp claims that every point x is both exclusive and exhaustive with respect to logic's truth and falsity operators—every x is such that exactly one of $\dagger A$ and $\neg A$ obtains for every sentence A in the language. Without this constraint the falsity clause is not superfluous.

Absent the given 'classical constraint' on logic's space of possibilities the truth/falsity conditions for logic's falsity (similarly truth) operator are completely compatible with the existence of glutty and/or gappy possibilities. After all, let A be both true-in-x and false-in-x. Problem? No; the truth/falsity conditions simply require that logic's falsity operator (viz., logical negation) also be glutty in x, that is, that $\neg A$ also be both true-in-x and false-in-x. The standard truth/falsity conditions simply don't rule out such glutty possibilities—unless, as per the classical course, one ad hocly rules them out of logic's space of options. And the same goes for gaps. Let A be neither true-in-x nor false-in-x. Problem? No; the truth/falsity conditions simply require that logic's falsity operator also be gappy in x, that is, that $\neg A$ also be neither true-in-x nor false-in-x. The standard truth/falsity conditions simply don't rule out such gappy possibilities—unless, per the classical course, one ad hocly rules them out of logic's space.

There is an apparent ad hocness in the classical constraint against the glutty and gappy possibilities recognized by logic on the FDE account. Some (of the vast number of) proponents of the standard account of logic might say that there's no such ad hocery; they might say that such alleged possibilities are ruled out by the very meaning of logic's falsity operator (i.e., logical negation). To such a claim comes an obvious question: what is the 'meaning' that does such a job?

I am no expert in meaning theory, and so cannot say much on the matter. What I can say is that if the meaning of logic's falsity operator is given in its truth/falsity conditions, then it looks—contrary to the running meaning-rules-them-out response to apparent ad hocery—that the meaning doesn't rule out gluts/gaps. If, on the other hand, the meaning of logic's falsity operator is something else—perhaps standard so-called operational rules for the operator (in some standard proof theory)—then the claim requires prior argument. Pending such argument, the ad hocery charge stands as a prima facie motivation against the standard account in favor of the FDE account; the former carries an ad hocery that the latter avoids.[27]

5. A Parting Gesture towards 'Inference'

The simple argument now sits above (viz., §4); and that completes the principal task of this paper. My hope is that with the simple argument plainly on the table, its virtues and vices may be drawn out in fruitful debate.

I'd like to close the paper with a gesture towards the important but messy question of rational inference—or rational 'change in view', or rational 'acceptance-rejection behavior', etc.—and FDE as logic.

Some (many?) philosophers look at the failure of material detachment (material modus ponens) in logical vocabulary as a radical or bizarre or otherwise implausible idea. Accordingly, that FDE is logic inherits the appearance of radicalness, bizarreness, and implausibility. But this is a bad response to the simple argument for FDE. As explained, we lose no true theories and gain valuable contenders for true theories; and material detachment is valid in such theories—not *logically* valid, but valid according to the theory's extra-logical closure relation (beefed up by shrieking and shrugging). What, then, is so radical? Nothing, as far as I can see.

But there is something that would be radical, bizarre or otherwise implausible; and that's the thought that our *default inference patterns*—our default acceptance-rejection patterns, default change-in-view patterns—don't in fact tend to involve a rejection of gluts (and/or gaps) as an initial first-go. Except in the face of phenomena that are themselves 'radical', 'bizarre' or so on (witness, again, the strange paradoxical or strange metaphysical-cum-theological phenomena), we *infer* in ways that underwrite the shrieking and/or shrugging of our theories. And a high degree of inductive support backs such default inference patterns: most of our true theories are classically closed, and we've bumped against insufficiently recalcitrant data to back away from classical closure of such theories. If the view according to which logic is FDE requires a rejection of such default patterns of acceptance-rejection behavior—a rejection of the apparent default patterns of inference behavior that govern the construction of our (mostly classical-logic-conforming) extra-logical closure relations—then the view stands in need of much more argument than the simple argument at the heart of this paper. But the view doesn't require as much.

Rational inference that guides the construction of closure relations on our true theories is a messy business, unlike the resulting closure relations themselves—including the universal, basement-level one (viz., logic). One thing remains clear (though, of course, defeasible): a rejection of gluts and gaps—and the tendency towards classical-logic-conforming closure relations—is a fruitful practice. But taken too far—from many cases to all—is as fallacious an inferential step as it has always been. Logic allows many strange possibilities even if this world exemplifies few of the options. But few is not none.

Notes

1. But see some previous work in which some of the issues are discussed (Beall 2013; 2015).
2. In saying as much I remain Quinean about our theory of logic; it's just another theory. That logic plays an exceptional role among relations involved in our true theories is not to suggest that our theory of logic is somehow epistemologically exceptional.
3. Throughout this paper I will slide 'consequence relation' and 'closure relation' together, though strictly speaking by 'closure relation' I mean the closure relation (operator) induced by the given consequence relation in the usual way advanced by Tarski (1956), namely, that we have a consequence/closure operator *Cn* defined via a 'prior' consequence relation: $Cn(X) = \{A : X \vdash A\}$, where $\vdash$ is the 'prior' consequence relation. What makes this slide acceptable is not only the aim of efficiency of presentation; it's that I assume throughout that logical consequence in fact satisfies the usual properties (including so-called structural properties) that induce a genuine closure operator standardly understood (i.e., an extensive, increasing and idempotent operator). These details are important in the background of the picture discussed here but they're suppressed to keep the discussion both simple and short.
4. If epistemologists show that this is incorrect, then just go with another example.
5. For recent discussion of demarcating logical terms in standard ways see Gil Sagi's work (2017; 2018).
6. This account was motivated by concerns of 'relevance' (Anderson and Belnap 2017; Anderson, Belnap, and Dunn 1992) but those motivations are not relevant to present purposes, where universal closure is the key idea.
7. Logic's truth operator—the null operator that generates 'nullations' (as the falsity operator generates 'negations')—is redundant according to logical consequence; it's usually omitted from the standard boolean stock but I think it's useful to occasionally remind ourselves that we in fact have the full symmetric picture of both a truth operator and a falsity operator.
8. A broader set-set relation is heuristically useful in some contexts but logical consequence qua universal closure relation is a set-sentence relation. For heuristic value of set-set relations in the present context, see Beall (2013; 2015).
9. Strictly speaking, I'm sliding between truth/falsity and satisfaction/anti-satisfaction conditions, but for present purposes the slide should be safe enough—especially since I will restrict the definition of consequence to sentences.
10. In what follows, a *u-variant* is per usual, namely, where u is any variable, and v' and v are variable assignments: v' is a u-variant of v just if v' differs from v only at u if anywhere.
11. For simplicity we define consequence only over the sentences, even though wff in general have semantic values (at a point, etc.).
12. An equivalent but different (and non-standard) approach is to keep *exactly* the same set $\wp(\{1, 0\})$ for values but restrict the account of 'counterexample' to avoid $\mathsf{b} = \{1, 0\}$ and $\mathsf{n} = \emptyset$.
13. If one prefers the lattice-driven picture, simply kick out b and n and draw a straight up-down line from t to f, and then the 'algebraic truth/falsity conditions' are the same.

14. But notice that if you kicked out all of logic's gappy models—focusing only on the restricted space of FDE models where dt2's antecedent holds for some A and B—and so maintained only the space of FDE 'possibilities' modeled by {t, b, f} then dt2 fails. In that space of options one has (for example) the 'necessary truth' (i.e., true at all such models in the restricted space) of $(A \wedge \neg A) \supset B$ even though in the given space there are counterexamples to $A \wedge \neg A \therefore B$ (viz., models where A is a glut and B has value f).
15. And if you rule out just one—as so-called Strong Kleene (Field 2008; Kleene 1952; Maudlin 2004) or LP theorists (Asenjo 1966; Beall 2009; Priest 1979; Priest 2006) do—the resulting account has a striking imbalance or asymmetric appearance in addition to apparent ad hocness (Beall 2017).
16. One may include in familiar De Morgan behavior double negation if, as Beall (2017) and Paoli (2015) discuss, this is essentially a De Morgan pattern of converting the dual of something—in this case, negation—into its dual (in this case, nullation or truth operator), much as logic's falsity operator converts the dual of binary connectives into the dual of duals, etc.
17. For a pioneering philosophical discussion of detachment, though centered mainly on Graham Priest's particular glut-theoretic approach to paradox, see Laura Goodship's (1996) work.
18. This is one of the classical validities whose validity (in effect) turns on the (alleged) logical truth of $A \vee \neg A$, but I give it explicitly here as the salient dual of 'Explosion' above.
19. Even though the 'relevance logic' motivations are irrelevant (as far as I can see) to logic qua universal closure— which is the key focus of the simple argument below—I hope that the simple argument is at least compatible with if not directly in line with what I take to be the spirit (and maybe even certain folk-lore slogans) of early Australasian 'relevant-logic' logicians (Routley et al. 1982).
20. Let P be unary for simplicity (and entirely without loss of generality).
21. Recall that we are taking logic to be FDE. The claim, of course, doesn't hold for some arbitrary account of logic qua universal closure.
22. Let P be unary for simplicity (and entirely without loss of generality).
23. Recall, again, that we are taking logic to be FDE. The claim, as in the dual shriek case, doesn't hold for some arbitrary account of logic qua universal closure.
24. A theory is prime just if it contains a (logical) disjunction iff it contains one of the disjuncts. The parenthetical '(logical)' in '(logical) disjunction' covers the possibility of a theory having various disjunction connectives in addition to logic's disjunction connective, which is in the language of all (true and complete-as-possible) theories.
25. And consider other similarly strange phenomena: property-theoretic paradoxes (e.g., 'I'm exemplified by all and only the properties that are exemplified by nothing' or 'I'm exemplified by something') or even, moving in other metaphysical directions, very strange entities such as gods—e.g., the omnigod or god-human figure in certain standard strands of Christian theories (theologies) of gods. These just look, prima facie, either glutty or gappy.
26. And for an argument from the leading subclassical accounts (viz., LP and K3) to the weaker FDE see Beall (2017).

27. Let me be clear that I present this 'bonus' argument for genuine debate; however, I do not see the argument to be as strong as the main no-loss-but-some-gain argument.

    For some of the complexity of meaning theory and its potential bearing on logical theory, see Gillian Russell's (2008) work.

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